FORMULATION OF THE OPTIMUM REPAIR POLICY PROBLEM WITHIN THE FRAMEWORK OF SYSTEMS ENGINEERING

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Abstract. This paper demonstrates how the concept of the value of the service provided by a system can be efficiently handled by the systems engineering methodology. Following a general formulation of the problem, two examples from the class of systems in which each element contributes to the system output are provided; the first to illustrate the allocation of repair effort to elements of unequal size, the second to show how the value placed on the output level influences the repair policy.

INTRODUCTION

Every system has a purpose and produces something. This is its **service**, and that service is deemed to have a **value**, W (measured e.g. in dollars per unit time). Due to (random) failures of its elements and subsequent repair actions, the system will settle down to a steady state in which the value is somewhat less than that of the fully intact state. The higher the rate of repair, and therefore also the **rate of repair cost**, C, the greater will be the value, and the general problem of the optimal repair policy is to maximize W-C by choosing the appropriate *allocation* of C.

This problem, and closely related ones, have been treated in several different ways, as is illustrated by a few representative references [1-3]. Much of this work has been concerned with developing rules or algorithms for how to allocate costs to the various repair elements and with how to manage them in order to optimize the cost-effectiveness of systems. These are both important and complex problems, as they influence many aspects of systems, such as training levels, maintainability, and spare parts scaling, but the formulation is usually in terms of specific cases and does not provide a basis for a common approach to such problems.

The purpose of the present paper is to show how the general problem can be elegantly formulated within the systems engineering framework [4-6] and to illustrate this approach by means of two examples within a particular class of systems.

GENERAL FORMULATION OF THE PROBLEM

Consider a system consisting of N interacting elements. The state of an element is described by a single binary variable $s_i \in \{0,1\}$, $i=1, ..., N$, corresponding to failed and operating, and the transitions between these two states are governed by a **failure rate, λ_i**, and a **repair rate, μ_i**.

This system has 2^N basic system states, each one characterized by an N-dimensional vector s with components s_i. The number of system states with respect to the two functional parameters W and C will depend on the extent to which the elements can be distinguished by these parameters. If the system is homogeneous (i.e. the elements are all identical), there are up to $N+1$ system states, characterized by the number, n, of operating elements. If the system is heterogeneous, there can be anything up to 2^N system states. In any case, let the system state space, S, consist of M states, with $2 \leq M \leq 2^N$, and let, as usual [6], $q(s_i) = q_i$, $i=1, ..., M$, satisfying the condition

$$\sum q_i = 1 \,, \qquad (1)$$

be the probability of finding the system in the state s_i; the vector $q = (q_1, ..., q_M)$ is the **superstate** of the system.

Transistions between system states, i.e. the dynamics of the system, are governed by the **transition rate matrix,** γ. This matrix can be written as the sum of three matrices. Two of these are the **failure rate matrix,** λ, and the **repair rate matrix,** μ, such that λ consists of the transition rates from a state to states with lower values of W, and μ consists of the transition rates from a state to states with higher values of W. The third matrix is a diagonal matrix with elements defined by

$$\gamma_{ii} = 1 - \sum \lambda_{ij} - \sum \mu_{ij} \,.$$

This simply ensures that the matrix γ satisfies the condition

$$\sum \gamma_{ij} = 1, \; i=1, ..., M \,, \qquad (2)$$

a necessary and sufficient condition for γ to be a transformation which preserves the probability character of q, as expressed by Eq.1.

A particular representation of the transistion rate matrix occurs if the system states are ordered such that $i>j$ implies $W(s_i)>W(s_j)$. Then λ is a lower triangular matrix and μ an upper triangular matrix, as shown in Fig.1. In the following, this ordering shall be assumed, and the corresponding representation of γ may be called its **normal** representation.

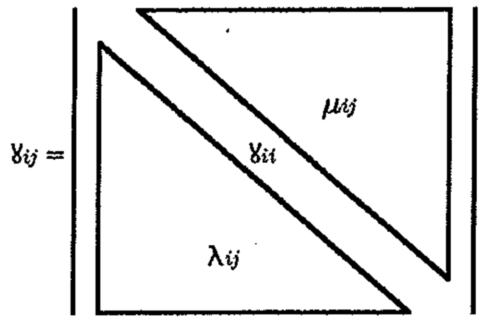

Fig.1 The decomposition of the transition rate matrix under its normal ordering.

In general, q is a function of t, and this can be made explicit by the notation $q(s;t)$. However, of particular interest is the case where q is not a function of t, i.e. $dq/dt = 0$; such a **steady state** is characteristic of many practical systems for the greater part of their operational lifetime. The condition that must be fulfilled in order for q to represent a steady state is

$$q\gamma = q \,. \qquad (3)$$

Consider now the case where the failure rate matrix λ is assumed to be given and unchageable, then Eq.3 can be viewed as a mapping from the set of all possible repair rate matrices, μ, to the set of superstates. That is, to each choice of μ there corresponds a steady state, q_μ, and consequently also a value, W_μ. But to each possible choice of μ there also corresponds a value of C_μ, the cost of providing the level of repair represented by μ, so the quantity $W_\mu - C_\mu = (W\text{-}C)_\mu$ is a function of μ, and there must, in principle, exist a particular choice of $\boldsymbol{\mu}$ which maximizes this function.

In the above, the repair rate matrix $\boldsymbol{\mu}$ is constrained to be a point in the set of *possible* repair matrices. What is possible is a practical problem, determined by the details of the case under consideration, and cannot be described further in any theoretical, generally valid form. In a particular case, different allocation of repair cost, e.g. to spare parts holdings, repair crews on standby, etc. will result in different repair rates, but there will be strong dependencies between the elements of $\boldsymbol{\mu}$.

A SIMPLE EXAMPLE

Consider a system consisting of two elements which have the same function and operate in parallel, as in the case of two generators supplying power to a common load. Element 1 has a capacity A_1, failure rate λ_1, and repair rate μ_1; element 2 has a capacity A_2, failure rate λ_2, and repair rate μ_2. The system has four system states,

SYSTEM STATE	ELEMENT 1	ELEMENT 2
4	1	1
3	1	0
2	0	1
1	0	0

The state diagram is shown in Fig.2, and the transition rate matrix, γ, is given by

$$\begin{matrix} 1-\mu_2-\mu_2 & \mu_2 & \mu_1 & 0 \\ \lambda_2 & 1-\lambda_2-\mu_1 & 0 & \mu_1 \\ \lambda_1 & 0 & 1-\lambda_1-\mu_2 & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & 1-\lambda_1-\lambda_2 \end{matrix}$$

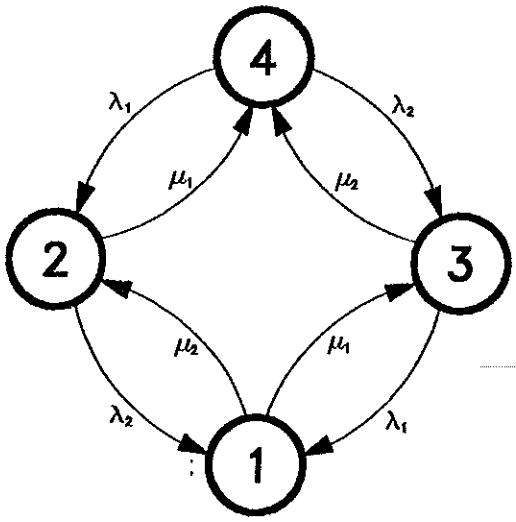

Fig.2 State diagram of the two-element system.

Consequently, the failure rate and repair rate matrices, introduced above, are given by, respectively,

$$\begin{matrix} 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \end{matrix}$$

and

$$\begin{matrix} 0 & \mu_2 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & 0 \end{matrix}$$

The superstate $q = (q_1, q_2, q_3, q_4)$ representing the steady state of this system is found by solving the four simultaneous linear equations implicit in Eq.3, and the result is:

$$q_1 = \lambda_1\lambda_2/[(\lambda_1+\mu_1)(\lambda_2+\lambda_2)] ,$$

$$q_2 = \lambda_1[1-\lambda_2(\lambda_2+\mu_2)]/(\lambda_1+\mu_1) ,$$

$$q_3 = \lambda_2[1-\lambda_1(\lambda_1+\mu_1)]/(\lambda_2+\mu_2) ,$$

$$q_4 = \mu_1\mu_2/[(\lambda_1+\mu_1)(\lambda_2+\mu_2)] .$$

To make this example really simple and easy to comprehend (but retaining the essential features it is supposed to demonstrate), it is necessary to make a few asumptions and approximations. The first of these is the assumption that the two elements have the same dominant failure mechanism and that this is independent of size, i.e. that $\lambda_1 = \lambda_2 = \lambda$. Introduce the two new variables,

$$\alpha = 2\lambda/(\mu_1+\mu_2) , \qquad \beta = (\mu_2-\mu_1)/(\mu_1+\mu_2) ,$$

and assume that $\alpha << 1$, then the superstate is (approximately) given by

$$q_1 = \alpha^2/(1-\beta^2) ,$$

$$q_2 = \alpha/(1-\beta) ,$$

$$q_3 = \alpha/(1+\beta) ,$$

$$q_4 = 1-2\alpha .$$

Normalize the element size by setting $A_1 = 1$, let $A_2/A_1 = x > 1$ (this restriction is simply a matter of the numbering of the elements), and assume that the cost of maintenance is proportional to α, but independent of β. As a result,

$$\partial(W-C)/\partial\beta = \partial W/\partial\beta .$$

All that is needed now is the value function, W, and for simplicity, W will be taken to be proportional to the system capacity. Then,

$$W = \alpha x(1-\beta) + \alpha(1+\beta) + (1-2\alpha)(1+x) ,$$

and now setting $\partial W/\partial\beta$ equal to zero yields

$$(1+\beta)/(1-\beta) = x^{1/2} .$$

This result is shown graphically in Fig.3, and is exactly what one would expect. With increasing difference in capacity between the two elements, the repair capability needs to be shifted more and more in favour of the element with the larger capacity.

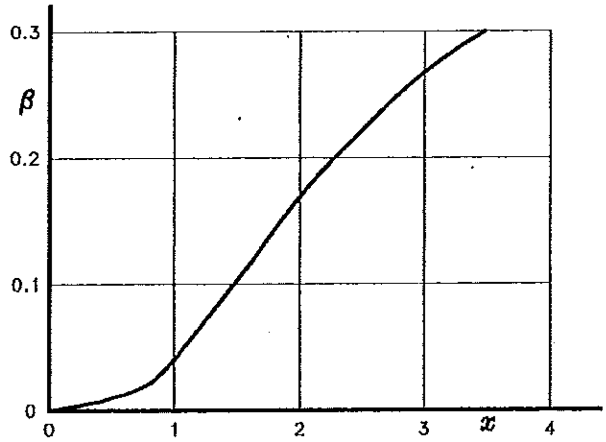

Fig.3 The split in repair rates, β, as a function of the difference in capacity, x.

A MORE GENERAL EXAMPLE

Consider the special class of systems where

a. the systems are homogeneous, i.e. all the elements within a system are identical;

b. each element is in one of two states, operating or failed, with transitions between them governed by a failure rate λ and a repair rate μ; and

c. each element, when it is in the operating stte, contributes equally to the magnitude of the system output, i.e. to the amount of whatever product or service the system produces.

The formulation of this definition is important. The elements are independent only with respect to the variable considered here (i.e. the magnitude of the output); it does not mean that the elements are non-interacting in other respects. For example, the dynamic stability of an electricity supply grid is very much dependent on the interactions between the generators, and the performance of an army depends on the interaction between its individual soldiers, but for many purposes one can equate output to the number of operating elements.

A system in this class which has N elements has $N+1$ system states, and the state diagram is shown in Fig.4. Its superstates are consequently $(N+1)$-dimensional

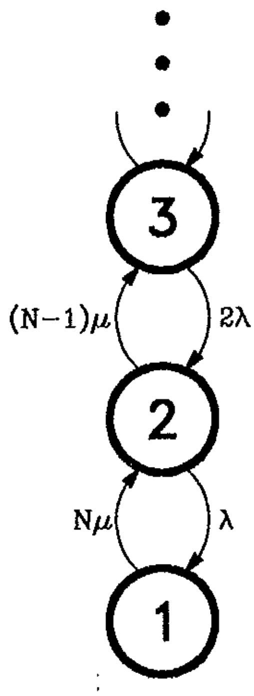

Fig.4 State diagram of the N-component system.

vectors, $q = (q_1, \ldots, q_{N+1})$, with binomially distributed components, so that

$$q_n = \binom{N}{n-1} (\lambda/(\lambda+\mu))^N (\mu/\lambda)^{n-1} ,$$

$$n = 1, \ldots, N+1.$$

Or, if the unit of time is chosen so that $\lambda=1$,

$$q_n = \binom{N}{n-1} (\mu+1)^{-N} \mu^{n-1} .$$

Assume that, in the range of interest, the rate of cost of maintenance, C, is linearly dependent on μ, i.e.

$$C = c_o + c\mu .$$

There then remains to be determined only the value function, and in order to demonstrate the importance of the value concept, three cases will be considered, as illustrated in Fig.5. In the first case, W is taken to be proportional to the system capacity, i.e.

$$W_n = w(n-1) .$$

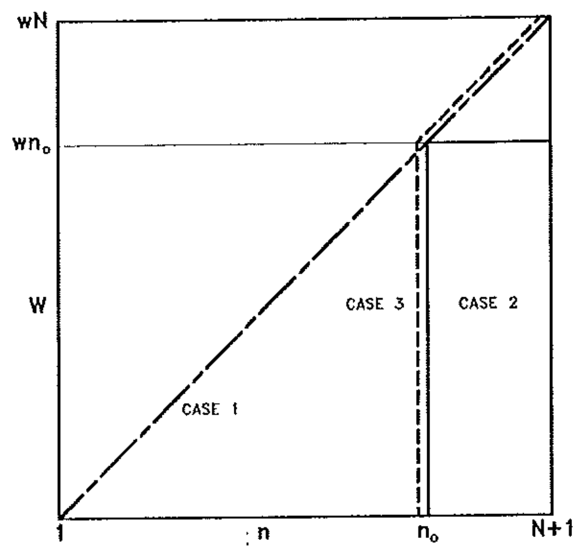

Fig.5 Three different value functions for the N-component system.

In the second case, W is given by the function

$$W = \begin{cases} n_o w, & n \geq n_o \\ 0, & n < n_o \end{cases},$$

and in the third case

$$W = \begin{cases} w(n-1), & n \geq n_o \\ 0, & n < n_o \end{cases}.$$

The solutions for the optimum value of μ as functions of c/w, with N and n_o as parameters, are not analytical expressions, so they have to be illustrated by means of some represenative numerical results. Figure 6 shows Case 1 (where the result does not depend on either N or n_o) and Case 2 for $N = 3,6,12,22$ and $n_o = N-1$.

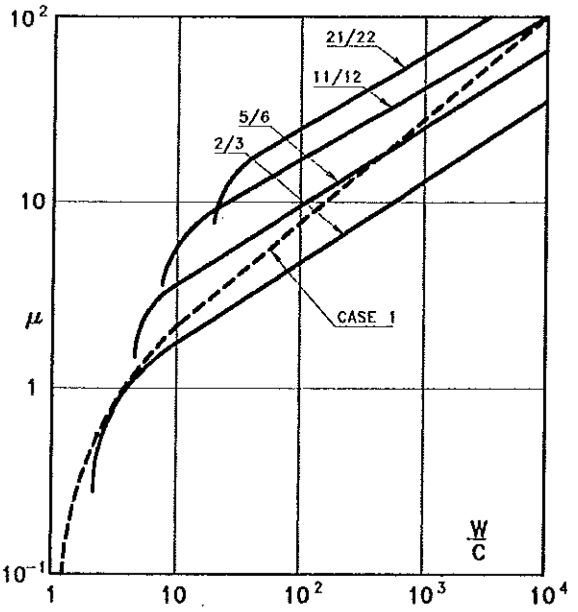

Fig.6 The optimum MTTR, μ, as a function of the value of the output of an element, w, for Case 1 and four instances of Case 2.

These curves show all the intuitively expected features:

a. As the value of the output increases, it pays to increase the repair rate (and cost).
b. This increase in the repair rate is less in Case 2 than in Case 1 because of the redundancy.
c. The optimum value of μ increases with increasing N because the W distribution and the q distribution need to match each other.
d. Below a certain value of the output, there is no optimum; it essentially does not pay to repair the system at all.

Figure 7 shows the influence of the degree of redundancy for Case 2 and a system with 12 elements. Increasing degree of redundancy means that the optimum repair rate is reduced, as one would expect.

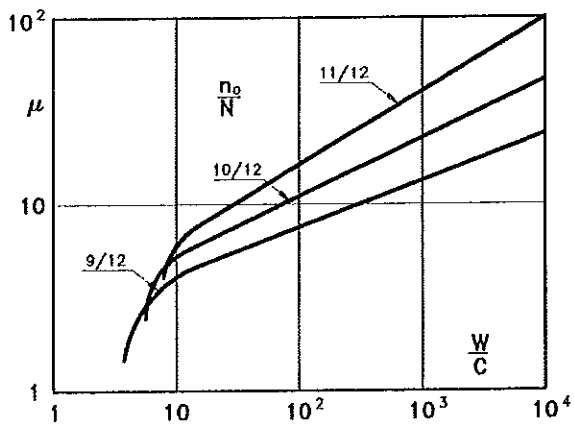

Fig.7 The influence of the degree of redundancy.

Finally, Fig.8 shows the difference between Case 2 and Case 3 for a system with six elements. It illustrates the fact that what may seem a small difference in value function can lead to a not insignificant change (e.g. a factor of 2) in the value of the optimal repair rate.

CONCLUSION

The optimum repair policy problem has been used to demonstrate the effectiveness of the systems engineering methodology and the central role played by the value concept within such a top-down methodology. Without a complete definition of the value of the service provided by a system, no meaningful optimisation of the system design is possible.

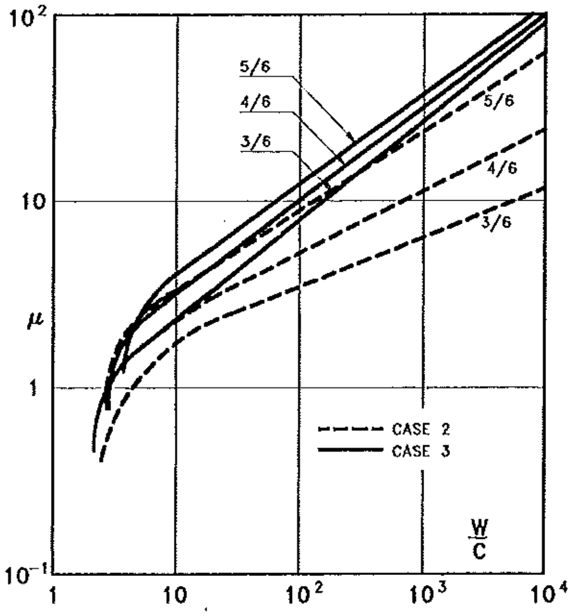

Fig.8 The influence of the detailed shape of the value function.

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