System Thermodynamics and the Concept of a System Temperature

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A simple, self-sustaining system, exhibiting the life cycle phases of growth, survival, and decay, introduced in a previous paper, is investigated further by means of an improved numerical model for handling the stochastic aspects. The system does not make any assumptions about its purpose, and only very general assumptions about the structure of the system and the processes involved in growth and decay. The central feature of the system is that the interaction between the system and its environment depends on the match between the structures of the system and the environment.

The motivation for both the choice of this particular system as representative of a significant class of systems and the approach to the investigation lies in the similarity between systems engineering and thermodynamics, and in the manner in which the purely phenomenological view of thermodynamics is explained in greater detail by statistical mechanics.

The main outcomes of the investigation are the importance of fluctuations, the role played by the concept of a system temperature, and the ability of the system to display a characteristic of living systems - existence within a relatively narrow temperature range only.

1 INTRODUCTION

1.1 Purpose of the Work

The point of departure for the work reported on here is the realisation that most of the objects created by engineers are required to remain operating for a period of time in order to repay the investment inherent in their creation, and that this in turn means that they have to be maintained to a greater or lesser extent during this period. From this perspective, it is an obvious step to consider the object (in the narrow, functional sense) and its logistic support together as an entity; as the true or complete object of engineering. Such an object is able to sustain itself through its interaction with its environment, and as this is a major characteristic of living organisms, we can consider such an object to be alive. The "alive" part comes, of course, and for the time being at least, from the involvement of people in the activities of this extended object, and our concept of an engineered object extends from a large hardware/software system with a few people in support to a pure organisation, where all the functions are carried out by people and hardware/software plays only an enabling role. (This latter type of system is often called an *enterprise*, with a corresponding subset of systems engineering, *enterprise systems engineering*.)

A further motivation for this work is the close *conceptual* affinity between systems engineering and thermodynamics, both being concerned with entities consisting of large numbers of interacting elements. Therefore, as thermodynamics is a relatively mature branch of science and systems engineering is a young discipline, it is natural to try to exploit the conceptual similarities with the aim of advancing the theoretical foundations of systems engineering.

Of course, the objects considered by thermodynamics are very different to those considered by systems engineering. The archetypical thermodynamic object is a (dilute) gas, whereas a typical object in systems engineering would consist of hardware, software, and people. In thermodynamics the number of elements is typically of the order of 10^{23}, whereas in systems engineering the breakdown into elements would typically stop before reaching 10^4. In thermodynamics, the interactions between the elements are normally relatively simple, ranging

from elastic collisions in a gas to the interactions between molecules in a living cell, whereas many of the objects considered by systems engineering involve interactions between humans. In a somewhat simplified summary, we can say that in the objects of thermodynamics the complexity arises from the number of elements involved, whereas in system engineering the complexity lies mainly in the interactions between the elements. Nevertheless, the statistical approach used in the extension of thermodynamics into statistical mechanics can, with appropriate modifications, be used to great advantage in investigating the properties of systems.

The appropriate modifications arise mainly from the most significant difference between the objects of thermodynamics (or physics in general) and those of systems engineering, which is that while the former *exist* as parts of Nature, the latter are *created* by engineers for particular purposes. We study Nature in order to develop conceptual models of its properties and behaviour, and the success of such models is measured in terms of the extent to which they are *true*. In engineering, success is measured in terms of the extent to which the engineered object meets the *intent* of its designer.

The purpose of this work is to develop a model that displays the essential behaviour of a self-sustaining system in a top-down fashion. A first attempt at such a development was presented in an earlier paper (Aslaksen, 2004), and the work reported on here is a continuation of that earlier work, using an improved mathematical model of the system. In particular, the motivation for choosing the main features of the system and the references provided in that regard there are just as relevant to this work, and need not be repeated here. However, for the reader's convenience, the definitions of the system and its parameters, as well as the existence of an equilibrium state, are repeated briefly in the two following subsections.

1.2 System Definition

The system consists of n identical *elements* that are able to establish interactions through *ports*; each element has m_0 ports. The interaction between two elements is initiated by what we shall call an *encounter*. The outcome of an encounter is one of two possibilities:

> *Formation* of a *link* between the two elements involved in the encounter, with probability p, and
>
> *dissolution* of a link attached to either of the elements involved in the encounter, with probability $(1-p)$.

If the two elements involved in an encounter are already supporting a and b links, respectively, then we shall assume that the probability of forming a link between them (i.e. the rate of link formation) is proportional to the number of unused ports on each of the elements involved, $(m_0-a)(m_0-b)$, and that the probability of dissolving an existing link (i.e. the rate of link dissolution) is proportional to the number of links existing on the two elements, $a + b$. Then the average number of links formed per unit time equals

$$\Phi_1(m_0 - \bar{a})(m_0 - \bar{b}), \tag{1.1}$$

and the average number of links dissolved per unit time equals

$$\Phi_2(\bar{a} + \bar{b}), \tag{1.2}$$

where Φ_1 and Φ_2 are two proportionality constants.

In the steady state, or equilibrium state, these two rates must be equal, which yields

$$\chi_0 = m_0 + \varphi - ((m_0 + \varphi)^2 - m_0^{\,2})^{1/2.} \tag{1.3}$$

with

$$\varphi \equiv \frac{\Phi_2}{\Phi_1}, \text{ and } \bar{a} = \bar{b} = \chi_0 , \tag{1.4}$$

The scalar parameter χ_0 can be interpreted as the simplest possible characterisation of the structure. Let ω_I be the number of elements supporting *i* interactions in a particular structure and let *n* be the total number of elements in the system, then

$$\chi_0 = \frac{1}{n}\sum_{i=1}^{m_0} \omega_i i , \tag{1.5}$$

and this can be interpreted as a measure of the *complexity* of the system.

It is worthwhile to note that in this system the elements themselves do not define a preferred value of χ_0, as is, for example, the case of atoms in a molecule. The value defined by Eq. 1.3 is determined solely by the assumed effect of encounters.

Each port has a *failure rate*, λ, and an element with one or more failed ports is replaced by a maintenance process that has a rate of replacement, μ, for the system equal to gn^{α}, where α is a constant, and *g* is a function of the match between the complexity of the system and that of its environment,

$$g = g_0 e^{-\frac{(\chi_0 - u)^2}{v^2}} , \tag{1.6}$$

where *u* represents the complexity of the environment; *v*, the *tolerance,* is a measure of how critical the match between the complexities of the two structures is, and g_0 is the maximum value of *g*, which occurs when there is a perfect match. The Gaussian form of the relationship in Eq. 2.6 was chosen as the simplest analytic expression of the expected features of the dependence of *g* on (χ_0-*u*), i.e. maximum at 0 and gradual decay away from 0.

To choose a value for α, we expect that the rate of replacement is proportional to the extent of the interface between the object and the environment. If the object is spherical, the interface surface is proportional to $n^{2/3}$, if the object is planar, the interface length is proportional to $n^{1/2}$; a compromise value is therefore $\alpha = 0.6$.

Finally, a consequence of the failure and replacement process is that the average (over all elements) of the intact ports, *m*, may be less than m_0, as will be discussed in Sec. 2.

1.3 Equilibrium State

The concept of an equilibrium state depends on describing the system in terms of variables that vary slowly enough to neglect the effect of terms proportional to d/dt in the equations relating them to each other. In particular, this means neglecting the fluctuations arising from the fact that the processes involved - decay and repair - are stochastic processes and working with the average values only. That is the basis of classical thermodynamics, and it will be the basis of our initial investigation of the system.

If we for a moment assume that *n* is so large that the function *n*(*t*) can be sensibly approximated by a continuous function, then we can easily display the time dependence of *n* for the case when $\mu \geq nm_0\lambda$. The behaviour is governed by the differential equation

$$\frac{dn}{dt} = gn^{\alpha} - m_0 \lambda n \tag{1.7}$$

which, by substituting $n^{1-\alpha}$ = u turns into a linear differential equation for u, and the solution (after reconverting back to n) is

$$n = \left[\frac{g}{m_0 \lambda} - (\frac{g}{m_0 \lambda} - n_0^{1-\alpha}) e^{-m_0 \lambda t} \right]^{\frac{1}{1-\alpha}} , \tag{1.8}$$

where n_0 is the initial value of n. The system behaviour displayed by Eq. (1.8) is one of a decreasing growth rate, with n converging towards a steady state, or *equilibrium* value n_∞

$$n_\infty = \left[\frac{g}{m_0 \lambda} \right]^{\frac{1}{1-\alpha}} . \tag{1.9}$$

This relationship between n and the other system parameters (with g determined by Eq. (1.6) and χ_0 determined by Eq. (1.3)) is the *state equation* for this system. For our assumption to be valid, the condition $g >> m_0 \lambda$ must hold; however, because n is actually an integer, growth will stop after a certain time, and this time divides the life span of the system into two *phases*, the *growth phase* and the *survival phase.*

To conclude this review of the system already presented in (Aslaksen, 2004), we note that there are *two* processes involved in maintaining the equilibrium state; the *internal* process, which maintains the order (structure) in the set of elements, and the *external* process, which is responsible for the interaction with the environment.

2 STABILITY OF THE EQUILIBRIUM STATE

2.1 The Maintenance Concept

The equilibrium state is reached at the transition between the growth and survival phases, and if there are no changes to the system or its environment, the survival phase and the equilibrium state will go on forever. However, just as in classical thermodynamics, the question arises as to how *stable* this equilibrium state is. What happens if there are fluctuations or abrupt changes in the system environment? That is the subject of this section.

To investigate the equilibrium state (i.e. when growth stops), we need to decide on a maintenance concept, and an obvious choice is to replace elements in the order in which they become defective (i.e. when the first port fails); this assumption was not made (nor required) in the approximate treatment given in (Aslaksen, 2004). The situation is then that of the n elements, n_0 are intact (each with m_0 ports) and $x = n-n_0$ are in a defective state, queuing up to be replaced. This is illustrated in Fig. 2.1.

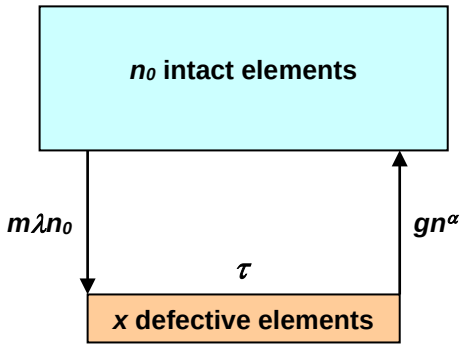


Figure 2.1 The transfer of elements between the set of intact elements and the queue of defective elements. The variable τ is the length of time a defective element spends in the queue.

Of the *x* defective elements arriving at the end of the queue, a proportion will have more than one failed port, and this proportion will increase with increasing time spent in the queue, τ. As the failure of ports is an independent process for each port, the average number of failed ports per element at the time of replacement is given by

$$\kappa(\lambda\tau, m_0) = m_0 - (m_0 - 1)e^{-\lambda\tau} \ . \tag{2.1}$$

Furthermore, we now have to allow for the possibility that changes to the environment could lead to further growth, and to describe the dynamics of the system, let

$$\delta = gn^{\alpha} - \lambda m_0 (n - x) \ ; \tag{2.2}$$

this is the net rate of creation of intact elements (i.e. $\delta = dn_0 / dt$), and these elements are used to replace failed ones (i.e. to decrease *x*) or, if $x = 0$, to increase *n*.

We then have three cases:

For $\delta < 0$

$$\frac{dx}{dt} = -\delta \ , \ \frac{dn}{dt} = 0 \ , \frac{dm}{dt} = \delta \ ; \tag{2.3}$$

for $\delta > 0$ and $x > 0$

$$\frac{dx}{dt} = -\delta \ , \ \frac{dn}{dt} = 0 \ , \ \frac{dm}{dt} = \delta\kappa \ ; \tag{2.4}$$

and for $\delta > 0$ and $x = 0$

$$\frac{dx}{dt} = 0 , \ \frac{dn}{dt} = \delta , \ \frac{dm}{dt} = 0 \ . \tag{2.5}$$

Here

$$\tau = \frac{x}{gn^{\alpha}} \ ; \tag{2.6}$$

$$g = g_0 e^{-\frac{(\chi-u)^2}{v^2}} \quad ; \tag{2.7}$$

$$\chi = m + \varphi - ((m+\varphi)^2 - m^2)^{\frac{1}{2}} \ . \tag{2.8}$$

2.2 Basic Numerical Calculation Schema

To investigate the system dynamics, i.e. to determine the functions *x(t), n(t),* and *m(t),* we have to resort to numerical integration, as there is no analytic solution to the set of coupled, nonlinear differential equations given above. To that end we use a numerical model, or *calculation schema*. Its basic and initial form is detailed below, and in subsequent sections it will be further developed to take into account the stochastic nature of our system. The choice of this schema determines the accuracy with which the numerical results represent the true (but unknown) behaviour of the system, and two features of the schema should be noted from the outset. Firstly, we shall treat *n* and *x* as real variables, whereas in reality they are, of course, integers. Besides being convenient, this reflects our "thermodynamic" approach of working with ensemble averages in equilibrium. Secondly, because of the nonlinearity introduced by the condition $x \geq 0$, the schema has to allow for a transition between the three cases defined in the previous section *within* a single time step.

In the equations of the previous section, n, m_0, g_0, u, φ, λ, and v are fixed model parameters; the variables κ, g, χ, and τ can be viewed as functions of x, n, and m. For convenience, denote the set of values of these variables for given x, n, and m by $\Psi(x,n,m)$, and let

$$\left.\begin{aligned} &\Delta t = t_{i+1} - t_i \ , \\ &\delta_i = \delta(t = t_i) , \\ &\Psi_i = \Psi(x_i, n_i, m_i) \ . \end{aligned}\right\} \tag{2.9}$$

The calculation schema, which might perhaps be called a half-improved Euler method, and which is the schema use throughout this investigation, is then as follows:

a. Calculate Ψ_i
b. Calculate the value of δ_i
c. Calculate next values of n and x:

$$\left.\begin{aligned} &\text{If } \ \delta_i \Delta t < x_i \ \text{ then} \\ &\qquad x_{i+1} = x_i - \delta_i \Delta t \, , \ n_{i+1} = n_i \, , \ m_{i+1} = m_i + \kappa \delta_i \Delta t / n_i \, ; \\ &\text{otherwise} \\ &\qquad x_i = 0, \ n_{i+1} = n_i + (\delta_i \Delta t - x_i) \, , \ m_{i+1} = m_i + \frac{1}{n_{i+1}} (\kappa x_i + 4(n_{i+1} - n_i)) . \end{aligned}\right\} \tag{2.10}$$

d. Calculate Ψ_{i+1}
e. Set $\Psi^* = (\Psi_i + \Psi_{i+1})/2$
f. Calculate new value for δ using Ψ^*, say δ^*
g. Recalculate the $(i+1)$-values for x, n, and m using δ^*

2.3 The Base Case

The calculation is carried out for the following choice of parameter values (this choice of what might be called a reference system or *base case* was justified in (Aslaksen, 2004)):

$$m_0 = 4 \qquad g_0 = 0.5 \qquad \alpha = 0.6 \qquad \lambda = 0.01 \qquad \varphi = 0.5$$

With this choice of values, the complexity of the system, χ_0, takes on the value 2.438. For the first calculation, let the initial complexity of the environment, u, equal this value. Then the equilibrium size of the system is $n_\infty = 552$, and there are no unrepaired elements in the queue, i.e. $x = 0$. Starting from this equilibrium state at some arbitrary time during the survival phase, $t = t_0$, let Δu be the value of the step taking place at t_0, such that $u(t > t_0) = u(t < t_0) + \Delta u$, then we find that the system response is sharply divided, depending on the value of the tolerance, v. For $\Delta u < \Delta u_0(v)$, the value of m (and, correspondingly, of χ) is reduced, but remains stable, and x takes on a corresponding value. (Note that the result shown in Table II of (Aslaksen, 2004), indicating that instability reappears for large enough negative values of Δu, was due to treating n as a stochastic variable.) But for $\Delta u > \Delta u_0(v)$, the value of m declines at an increasing rate and x increases to n_∞, representing a disintegration (or "death") of the system.

The result of a run of the calculation for $\Delta u = 0.01335$ and $v = 0.2$ is shown in Fig. 2.2. The value of x quickly reaches what would be its new equilibrium value of about 10, but then continues to rise slowly for a time period T until it rises quickly to n_∞. The time period T, in this case something on the order of 1,500 units of time, is very sensitive to the value of Δu; for $\Delta u = 0.01325$ is is on the order of 4,500 units of time, and for $\Delta u < 0.0132$ it is infinite; that is, the system remains in its new equilibrium state.

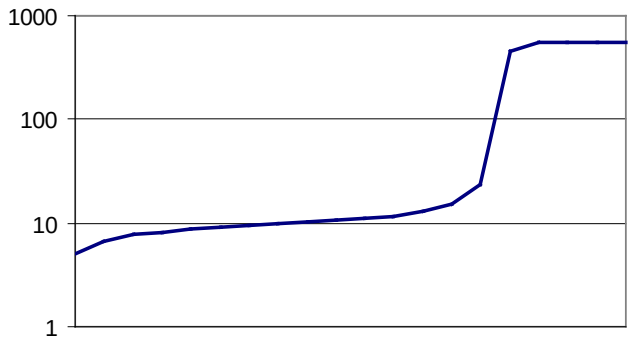


Figure 2.2 The function $x(t)$ for $0 < t < 2000$ units of time.

2.4 Varying the Initial Environmental Complexity

In the previous subsection, the initial condition was $\chi_0 = u$. Let us now investigate how the stability of the equilibrium state is affected if there is an initial mismatch between the system and environment complexity, i.e. $u = \chi_0 + \varepsilon$. In accordance with Eq. (1.6), the parameter g takes on a value less than g_0, and the equilibrium value of n is correspondingly reduced, in accordance with Eq. (1.9). The reduction is, of course, independent of the sign of ε, and for the base case it is as shown in Table 2.1:

ν/ε	0	0.01	0.02	0.03
0.2	552	549	539	522
0.5	552	552	550	547
1.0	552	552	552	551

Table 2.1 The equilibrium system size, n_∞ (rounded to the nearest integer), as a function of the tolerance, ν, and the complexity mismatch, ε.

Determining the limit of stability for each of these cases, i.e. $[\Delta u]_{max}$ as a function of ν and ε, results in the picture displayed in Fig. 2.3.

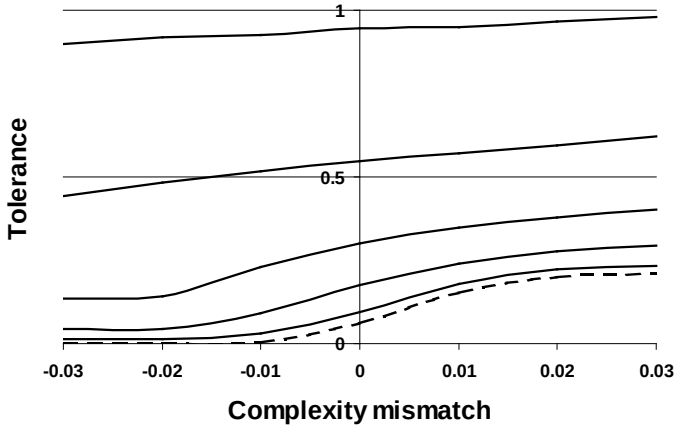
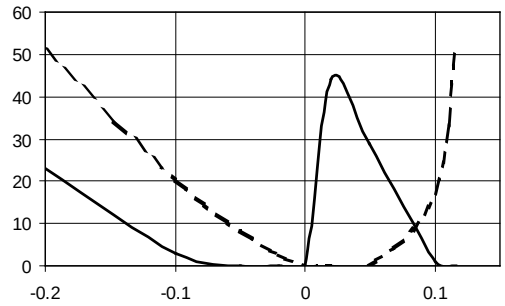


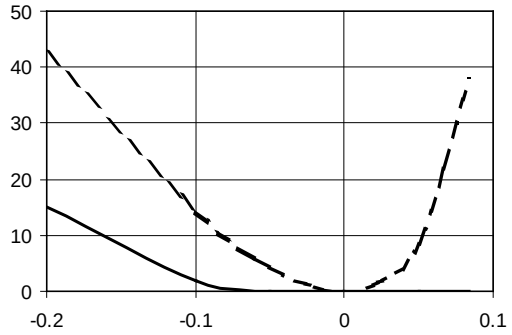
Figure 2.3 Curves of constant values of $[\Delta u]_{max}$ in the (ε,ν)-plane. From the top, the values are 0.3, 0.1, 0.03, 0.01, and 0.003. The bottom curve (dashed) indicates a very small value of $[\Delta u]_{max}$, say, less than 10^{-4}, as $[\Delta u]_{max}$ does not become zero in this plane (exept on the ε-axis), it just falls off very sharply near this bottom curve.

In order to have a clear understanding of how the system behaves under these abrupt changes to the complexity of the environment, Fig. 2.4 shows the effect of varying the magnitude of the step, Δu, for three values of the initial mismatch, ε, and with the tolerance, ν, chosen to be 0.5.

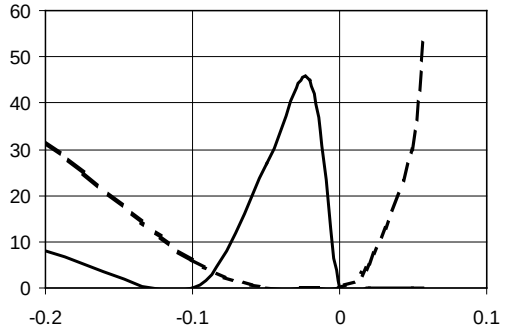
In Fig. 2.4, the solid line is the deviation of n_∞ from its value for $\Delta u = 0$, multiplied by 10, and in both (a) and (c) there is a peak when $\Delta u = -\varepsilon$. The value at the peak is 4.9, which is equal to the difference between the two values of n_∞ for $\varepsilon = 0$ and $\varepsilon = 0.03$. That is, the step in the complexity just compensates for the initial mismatch. The increase in n_∞ as Δu becomes increasingly negative arises from the effect of the decrease in m in Eq. (1.9), while g remains constant as the system reduces its complexity to match that of the environment. For increasing positive values of Δu there is no way the system can increase its complexity (under the present assumption of no fluctuations in the internal processes), and soon the mismatch becomes so great that instability occurs. The shape of the x-curves reflects this; x is an indirect measure of the decrease in system complexity due to the increase in failed ports.



(a) ε = -0.03.



(b) ε = 0.



(c) ε = 0.03.

Figure 2.4 Showing the dependence of n_∞ (solid line) and x (dash line) on the value of the step change in the complexity of the environment, Δu, for three values of the initial mismatch between the complexity of the system and its environment, ε (but see the text for a definition of the vertical scale).

3 CONSIDERATION OF THE STOCHASTIC NATURE OF THE VARIOUS SYSTEM PROCESSES

3.1 Stochastic Port Failure Process

So far, the model has been formulated in terms of average values; in the next step in the development of the model the stochastic nature of the port failure process will be taken into account. If at time $t = t_0$ there are n elements, each with m_0 intact ports, the average number of

ports to fail in the time interval $t_0 < t < t_0+dt$ equals $\lambda n m_0$. The probability that exactly z ports will fail in this time interval is given by the Poisson distribution,

$$f(z;\lambda n m_0) = \frac{(\lambda n m_0)^z e^{-\lambda n m_0}}{z!} \ . \tag{3.1}$$

In order to simplify the application of this distribution in the numerical calculations, Eqs. (2.2) and (2.5) are rewritten as follows:

$$\delta = \lambda n m_0 (1 - \frac{x}{n}) - g n^{\alpha} \ ; \tag{3.2}$$

and

$$\frac{dm}{dt} = \frac{1}{n}(g n^{\alpha} \kappa(\lambda\tau, m_0) - \lambda n m_0 \frac{m}{m_0}) \ ; \tag{3.3}$$

In these equations, the quantity $\lambda n m_0 dt$ is then replaced by the random variable z, and the distribution of this variable is given by Eq. (3.1).

Finally, the time step in the numerical integration, dt, is chosen so that the mean value of z equals 5; i.e. $dt = 5/\lambda n m_0$. For the reference system, dt = 0.226 units of time. Assigning a random value to z then means generating a random number in the range 0 - 1, and mapping it on the cumulative distribution of z, in the bands shown in Table 3.1. E.g., the value of z is 2 if the random number is less than or equal to 0.195, but larger than 0.083.

z	0	1	2	3	4	5	6	7	8	9	10	11	12
F'(z)	0.023	0.083	0.195	0.353	0.528	0.689	0.815	0.900	0.950	0.977	0.991	0.996	>0.996

Table 3.1 Values of the modified cumulative Poisson distribution used to generate random values of z.

The behaviour of the system when the stochastic nature of the element failure process is taken into account, is illustrated by a typical case in Fig. 3.1. The system parameters are those of the base case, with $\varepsilon = 0$ and $\nu = 0.2$, and the system size rises rapidly to n= 552 (the equilibrium value without fluctuation) and then with decreasing rate up to a value of about 800. This is the *growth phase*. The system then stays at this size for a while, and the value of m is all the time equal to its maximum value, $m_0 = 4$. This is the *survival phase.* But then the value of m suddenly starts to decay, which means that the system is disintegrating. The point at which $m = 0.75 m_0$ will be considered to be the end of the system life time, or the *death* of the system.

It is important to note that, while we shall speak of the system being in equilibrium during the survival phase, the system is in fact slowly decaying during this phase, and the "death" of the system is simply a very rapid increase in the rate of decay. This use of the term "equilibrium" is exactly the same as in equilibrium thermodynamics; the "dynamics" are restrained to keeping the system state "close" to equilibrium.

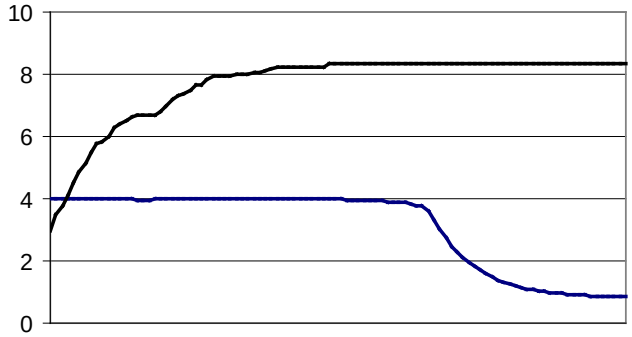


Figure 3.1 A typical behaviour of the system when the stochastic nature of the element failure process is taken into account, with the parameters of the base case, and $\varepsilon = 0$ and $\nu = 0.2$. The upper curve is the number of system elements (i.e. system size), divided by 100; the lower curve is the element integrity, m. The total duration of the graph is about 425 units of time, and the system life time is about two-thirds of that.

For given system parameter values, the shape of the function *n(t)* and the system life time vary considerably, due to the stochastic nature of the failure process. For the base case, with ε and *v* as variables, the average, T, and standard deviation, σ, of the life time was calculated over 1000 runs for each set of (ε,*v*)-values, and the result, in the form of curves of constant values of T in the (ε,*v*)-plane, is shown in Fig. 3.2.

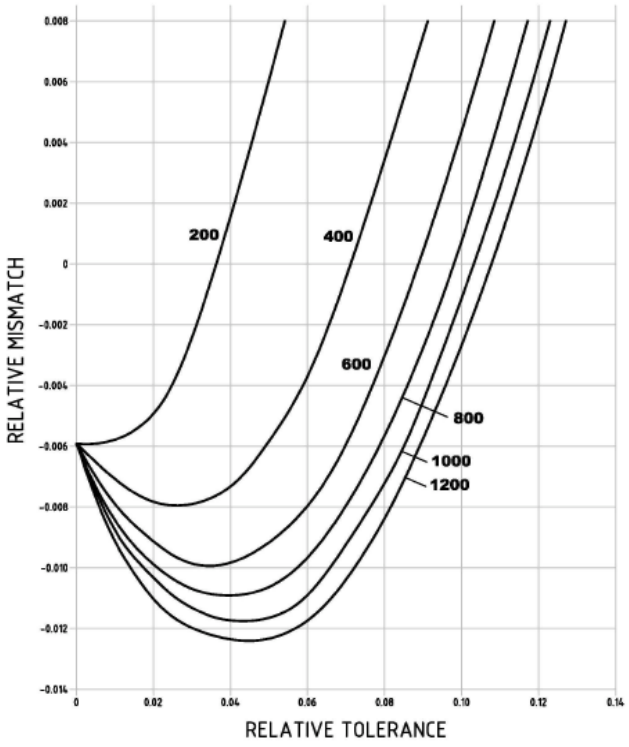


Figure 3.2 Curves of constant value of the system life time, T, in units of time, in the (ε,*v*)-plane.

In Fig. 3.2, the values of ε and *v* are given in relative terms, that is, relative to the nominal value of χ_0, which for the base case is 2.438.

The curves in Fig. 3.2 require some explanation. Firstly, while they are shown to come together at a point on the vertical axis, i.e. for $v = 0$, this is not a calculated point, but an extrapolation only. Reference to Eq. (1.6) shows that g diminishes rapidly towards zero as v gets smaller, and the smallest value of v that the numerical model can handle is about 0.015. The existence of such a point (or a steeply rising curve on the $v = 0$ plane) can be expected from the results displayed in Fig. 2.3, but the fact that the curves have a minimum for some value of $v > 0$ is surprising.

To explain this, we need to examine the behaviour shown in Fig. 3.1 more closely. What we had previously called the growth phase now consists of two distinct time periods; first a period of rapid growth, in which the fluctuations play no significant role, and then a period in which the fluctuations start to slow down the rate of growth, but also allow the size of the system (i.e. the value of n) to increase beyond the value of n_∞ when there are no fluctuations (i.e. 552 for the base case). This part of the growth period increases rapidly with decreasing value of v, whereas the duration of the survival phase decreases with decreasing value of v.

The determinant of this behaviour is the feedback mechanism provided by the dependence of the repair process on the match between the system complexity and that of the environment, as embodied in Eqs. (1.6) and (1.7). The fluctuations in the port failure rate, λ, leads to fluctuations in the number of failed elements, x, which equates to fluctuations in m, which, through Eq. (1.3), leads to fluctuations in the system complexity, χ, and finally to fluctuations in g, which then either stabilise (negative feedback) or destabilise (positive feedback) the system, as was demonstrated in Sec. 2.4. In Fig. 3.3, the function $g(\chi)$ is shown for two values of the tolerance v, and the fluctuations in χ are represented by the grey band, initially centred around $\chi_0 - u = -\varepsilon$. The reason for the band is that it is the fluctuations in g that compensate the fluctuations in λ.

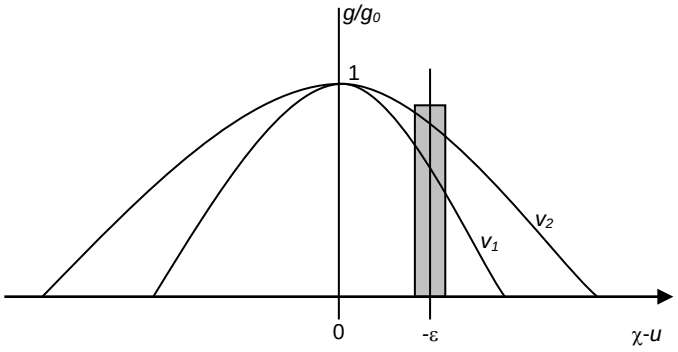


Figure 3.3 The function $g(\chi)$ for two values of v, with $v_1 < v_2$, and the with fluctuations in χ represented by the grey band.

If there were no fluctuations, the value of g would remain as determined by the value of the mismatch, ε, and decreasing the value of v results in a decreasing value of g and thereby of n_∞ (see Eq. (2.9)), as shown in Table 3.1. However, because n can only increase (when the failure rate fluctuates below its average value) and not decrease (when the failure rate fluctuates above its average value), the value of χ_0 will start to move towards u as soon as n becomes large enough to make $x > 0$ and thereby for the fluctuations in λ to result in fluctuations in χ. This movement continues until the left-hand side of the grey band reaches $\chi - u = 0$,; from this moment on there is an increasing probability of the fluctuations driving the state of the system into the left-hand side of Fig. 3.3 (i.e. $\chi - u < 0$), in which case the positive feedback drives the system complexity to zero. That moment is the transition between the two periods of the growth phase discussed above, and it explains why ε has to be below a certain value for the first period to be significant, as illustrated in Fig. 3.2.

The standard deviation is a function of T, and between T = 200 units of time and T = 1200 units of time, σ/T varies linearly from about 0.63 to 0.79. If we consider a value in the middle of this range, say, $\sigma/T = 0.7$ for T = 600 units of time, then the standard deviation of the distribution of means for our sample size of 1000 is about 12, so this is an indication of the accuracy of the results shown in Fig. 3.2.

3.2 Stochastic Ordering Process

As the next step in the investigation of the system behaviour, we shall take into account the stochastic nature of the internal process that maintains the links between the elements and thereby determines the structure of the system. Our approach to this is related to the approach taken in (Aslaksen, 2004), but instead of simply introducing a fluctuation amplitude, we shall define and look at the processes giving rise to these fluctuations in more detail. However, in order to restrict the scope of this particular part of our investigation, we will limit it to the base case, i.e. $m_0 = 4$, $g_0 = 0.5$, $\alpha = 0.6$, $\lambda = 0.01$, and $\varphi = 0.5$.

The first step is to consider the timescale involved. The unit of time is defined by the failure rate, λ, or, more properly, by its inverse, the Mean Time Between Failures (MTBF) of the ports. For example, in the base case, the unit of time is equal to 0.01*MTBF. The time step used in the integration of the differential equations, *dt*, was chosen to equal $5/\lambda m_0 n$ or, again for the base case, in the order of 0.2 units of time. For our assumption about the applicability of Eq. (1.3) to be valid, the number of links formed and dissolved in this time step must be very large. The average number of elements replaced during the time step is 5 (in the survival phase), each replacement requires an additional χ links to be created, or a total number of additional links of about 12. For this to be negligible compared to the number of links formed (and dissolved) during a time step, $\Phi_1(m_0 - \chi_0)^2 dt >> 12$ (see Sec. 1.2). Let us for the moment assume that this condition is satisfied and investigate what the effect of the fluctuations caused by the stochastic nature of the encounter process is; in Sec.4 we shall then consider what happens when the condition is not satisfied.

Referring back to Sec. 1.2, let the average number of encounters within the system per unit time, the *frequency of encounters*, be denoted by ν. The number of encounters within a time step *dt*, *z*, is given by

$$z = \nu\, dt\,. \tag{3.4}$$

The probability that such an encounter will lead to the formation of a link, *p*, is given by

$$p = \frac{(4-\chi)^2}{(4-\chi)^2 + \chi}\,, \tag{3.5}$$

and the probability that it will lead to the dissolution of a link is, of course, given by

$$1 - p = \frac{\chi}{(4-\chi)^2 + \chi}\,. \tag{3.6}$$

In equilibrium, the two probabilities are equal, and $p = 0.5$. But the formation and dissolution process is a stochastic process, and in a time period *dt* the change in the number of links in the system is typically as shown in Fig. 3.4, where, for simplicity, the encounters have been shown evenly spaced.

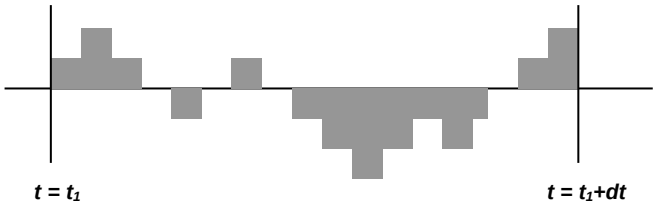


Figure 3.4 Formation and dissolution of links during one time step, *dt*.

The stochastic variable of interest is the average deviation of the system complexity, χ, from its equilibrium value due to the stochastic change in the number of links during a time step *dt*. Let this variable be denoted by $\Delta\chi$. Due to the mutual dependency of *p* and χ, as defined by Eq. (3.5), the distribution of $\Delta\chi$ can only be determined by numerical simulation, and in Fig. 3.5 the relative standard deviation, σ/χ_0, of the distribution of $\Delta\chi$ is shown as a function of the number of encounters in *dt*, *z*, for four values of the number of system elements, *n*.

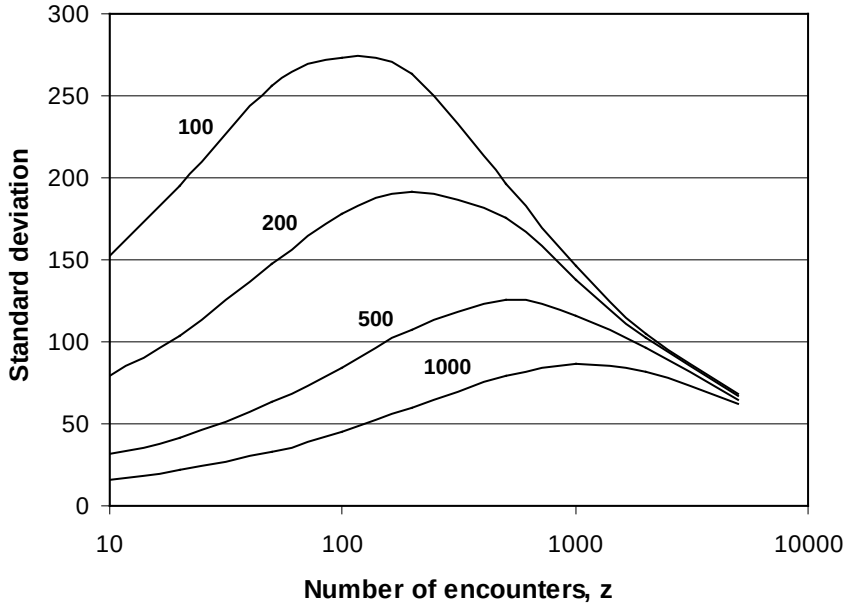


Figure 3.5 The relative standard deviation, σ/χ_0, in units of 10^{-4}, of the distribution of $\Delta\chi$, as a function of the number of encounters, *z*, in a time step, *dt*, for four values of the number of system elements, *n*.

The result in Fig. 3.5 can be understood as follows: If the non-linearity due to the dependence of the transition probabilities on χ is ignored, so that the probability of the formation of a link equals that of dissolution of a link, i.e. both equal to 0.5, then a simple numerical calculation shows that the average deviation of the number of links from its equilibrium value over the interval *dt* has a distribution with a variance proportional to *z*. That average is composed of the net area illustrated in Fig. 3.4 divided by *z*, so that the fluctuations in the net area alone has a $z^{3/2}$ dependence on *z*. Due to the non-linearity, this exponent decreases to about ½ as *z* increases; and as the fluctuations in χ are proportional to the fluctuations in the net area divided by the number of elements, *n*, the point of sharpest decrease, i.e. the peak of the curves in Fig. 3.5, occurs at a value of *z* proportional to *n*.

With this understanding, it is reasonable to attempt to approximate the *z*-dependence of the relative standard deviation, as shown in Fig. 3.5, by an expression of the form

$$\frac{\sigma}{\chi} = a\, e^{-\frac{z}{b}}\sqrt{z} + c\,(1 - e^{-\frac{z}{d}})\frac{1}{\sqrt{z}}\,. \tag{3.7}$$

This turns out to provide a good approximation (to within ±3 %), with the following dependence of the four parameters on *n*:

n	a	b`	c	d
100	0.0001409	704	0.4471	1301
200	0.0004598	508	0.4627	925
500	0.0008278	121	0.4561	269
1000	0.0015529	54	0.4575	126

Table 3.2 Dependence of the parameters in Eq.(3.7) on the number of elements in the system.

For the numerical calculations, we shall simplify the distribution of the fluctuations in χ to be rectangular, i.e. a constant probability density of 1/2σ for $-\sigma < \Delta\chi < \sigma$ and zero otherwise. A random value of Δχ in this range is added to χ before the final calculation of the increments in the variables (*n*, *m*, and *x*) during a step of duration *dt*. However, the calculations show that this additional fluctuation makes no change to the average system life time (within the accuracy of the numerical calculation), and it will not be considered in the calculations in Section 4.

In this calculation, it was assumed that the density distribution was symmetric around the equilibrium value of χ. Due to the form of the non-linearity that is not strictly the case, but the effect is negligible, as is easily demonstrated. For small values of Δχ (i.e. neglecting higher powers of Δχ in Eq. (3.5)), and for the base case, the probability of an encounter leading to the formation of a link, *p*, is given by

$$p(\Delta\chi) = \frac{2.438 - 3.124\cdot\Delta\chi}{4.876 - 2.124\cdot\Delta\chi}\,. \tag{3.8}$$

This function is shown in Fig. 3.6.

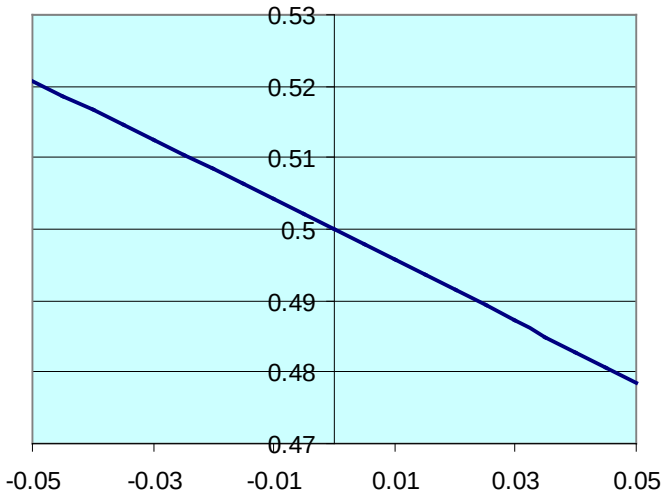


Figure 3.6 The probability *p* as a function of a small deviation, Δχ, of χ from equilibrium.

The curve in Fig. 3.6 is not a straight line; it has a slightly negative second derivative. Consequently, the integral

$$\int_{-\sigma}^{\sigma} (p(\Delta\chi) - 0.5)\, d(\Delta\chi) \tag{3.9}$$

is not zero, as it has to be in equilibrium, but is, for values of σ less than about 0.05, approximately equal to $8\cdot\sigma^3$. In order for the integral to be zero, the equilibrium value of χ has to shift by a small amount, ε, from 0.5, given approximately by $\varepsilon = 0.18\cdot\sigma^2$, which demonstrates that this shift is negligible compared to σ.

This is an important observation, because in the next section we shall see that a shift in χ does have a very significant influence on the behaviour of the system.

4 COUPLING OF THE INTERNAL AND EXTERNAL PROCESSES

4.1 Details of the Repair Process

In Sec. 1.2 we introduced a process that created new elements. These new elements were used to replace elements on which one or more ports had failed and, if there were any left over, to increase the size of the system. However, nothing was said about how these elements were reintegrated into the system by forming links to other elements. We shall now define this process as follows:

> *When a failed element is replaced by a newly created element, the new element takes over the links that were still maintained by the failed element as part of the replacement process.*
> *When a newly created element is added to the system, it has initially no links. The links have to be established through encounters.*

In Section 2.1 we found that, at the time of replacement, a failed element will have, on the average, a number of failed ports given by the function $\kappa(\lambda\tau, m_0)$, and if we use Eq. (1.3) to relate the number of links, χ^*, to the number of intact ports, $m_0 - \kappa$, we have (see also (Aslaksen, 2004))

$m_0 - \kappa$	1	2	3
χ^*	0.382	1.000	1.697

Table 4.1 The relationship between the number of intact ports, $m_0 - \kappa$, and the average number of links, χ^*, maintained by an element.

To within an accuracy of ±5 %, this relationship can be expressed by the following equation

$$\chi^* = 0.325(m_0 - \kappa) + 0.081(m_0 - \kappa)^2 \ . \tag{4.1}$$

The number of elements created per unit time is equal to gn^α (see Sec.1.2) and, when there is no increase in system size, the total number of links that have to be re-established through the process of encounters per unit time as a result of the repair process is therefore equal to $gn^\alpha(\chi - \chi^*)$. However, if a certain number of new elements, say sgn^α, are created, then $sgn^\alpha\chi$ links have to be created per unit time in addition to those that have to be created as a result of the repair process, which is now equal to $(1-s)gn^\alpha(\chi - \chi^*)$, so that the total number of links that need to be

created per unit time, μ, in order to integrate new and repaired elements into the system is given by

$$\mu = gn^{\alpha}(\chi - \chi^{*}(1-s)). \tag{4.2}$$

For this additional number of links to be formed, the probability of an encounter leading to the formation of a link, *p*, can no longer equal 0.5, but must be slightly larger, and the probability of an encounter leading to the dissolution of a link correspondingly smaller, so that the excess number of links formed per unit time is equal to ν(2*p* -1). This requires a small shift in χ, in accordance with Eq. (3.8), and using this expression for $p(\Delta\chi)$ and equating ν(2*p* -1) to $gn^{\alpha}(\chi - \chi^{*})$, we obtain (again, to better than ±3 % for $\Delta\chi < 0.1$)

$$\Delta\chi = -\frac{gn^{\alpha}(\chi - \chi^{*}(1-s))}{0.81 \cdot \nu}. \tag{4.3}$$

Equation (4.2) expresses the fact that our more detailed definition of the repair process results in a coupling of the two previously introduced processes - the *internal* process, which maintains the order (structure) in the set of elements, and the *external* process, which is responsible for the interaction with the environment. The parameter χ represents the internal process, and the parameter *g* the external process.

The significance of this result is that the net formation of links required by the repair process in order to integrate newly created elements into the system demands a shift in the value of the system complexity, Δχ, from what was previously considered its equilibrium value, as given by Eq. (1.3). In itself, this new definition of the equilibrium value of χ is not significant, because the complexity of the environment is a quantity we have been able to choose freely. What is significant is the coupling between *changes* in the rate of net formation of links and the *changes* in χ; this modifies the dynamics of the system and thereby its behaviour under fluctuations.

To illustrate this, Table 4.2 shows the effect of modifying χ by Δχ in the numerical model and varying the value of ν in Eq. (4.2) on the mean life time T and the mean size of the system, N, calculated for the base case.

ν	**200**	**500**	**1,000**	**2,000**	**5,000**	**10,000**
T	170	269	344	405	459	475
N	425	630	717	756	776	783

Table 4.2 Mean life time, T, and mean system size, N, as functions of the frequency of encounters, ν.

4.2 The Concept of a System Temperature

We have seen that the system equilibrium state existing during the survival phase is maintained through a balance between a failure process, characterised by the port failure rate, λ, and a repair process that involves the internal and external processes. The latter depends on the coupling between the system and its environment, expressed by the coupling function, *g*, which depends on the system complexity, χ, according to Eq. (1.6). The system complexity again depends on the balance between link formation and dissolution, according to Eqs. (1.3) and (4.2).

Thus, in effect, the equilibrium state can be considered to depend on two system parameters; the port failure rate, λ, and the net link formation rate. Drawing again upon the analogy with

statistical mechanics, we expect both of these parameters to be related to what we shall for the moment loosely call "the energy of an element" and, furthermore, we shall consider this to be the equivalent of a "free" energy that can be exchanged between elements in an encounter. In the present context we should think of the use of the word "energy" in a generalised sense, and what is exchanged may sometimes be better characterised as knowledge, information, manpower, spare parts, or even just money.

Furthermore, and again in analogy with statistical mechanics, we expect the energy of an element to be a stochastic function, and that our knowledge will be restricted to a distribution function $f(\alpha)$, such that the probability of any particular element having a free energy in the range α to $\alpha + d\alpha$ is given by $f(\alpha)d\alpha$. We can then define a *system temperature*, θ, by requiring that the mean energy of an element be proportional to θ, i.e.

$$\int_0^\infty \alpha f(\alpha;\theta)\, d\alpha = \kappa\theta, \tag{4.4}$$

where κ is an unspecified conversion factor from temperature to energy (the equivalent of Boltzmann's constant in thermodynamics), and with $f(\alpha;\theta)$ satisfying the usual constraints,

$$f(0;\theta) = f(\infty;\theta) = 0; \tag{4.5}$$

and

$$\int_0^\infty f(\alpha;\theta)d\alpha = 1. \tag{4.6}$$

The dependence of the net link formation rate on θ requires a further detailing of the formation and dissolution processes (and thereby, of course, a reduction in the size of the set of systems of which this system is an instance). First of all, let us extend the concept of an *encounter* between two elements to include any instance of an interaction (i.e. exchange of energy) between them, whether it leads to the formation or dissolution of a link or not. In fact, we expect that only a small fraction of the encounters per unit time, the *encounter frequency*, ν, lead to the formation or dissolution of a link, and our previous (simplified) concept of a probability p of an encounter leading to the formation of a link (and (1-p) leading to the dissolution of a link) is no longer valid.

Secondly, we now introduce two further system parameters, the *formation energy* ε_1 and the *dissolution energy* ε_2, and postulate that the probability of an encounter leading to the formation or dissolution of a link is proportional to the amount by which the expectation value of the sum of the free energies of the two elements involved in the encounter is greater than ε_1 and ε_2, respectively. The distribution of the sum of the two element energies is given by

$$f^+(\alpha;\theta) = \int_0^\infty f(\alpha';\theta)\, f(\alpha-\alpha')d\alpha\ , \tag{4.7}$$

and the proportionality factor is given by

$$\Phi'(\theta;\varepsilon) = \int_\varepsilon^\infty (\alpha-\varepsilon)\, f^+(\alpha)d\alpha\,. \tag{4.8}$$

To obtain the function $\Phi(\theta;\varepsilon)$ introduced in sec. 1.2, the proportionality factor in Eq. (4.8) has to be multiplied by the product of the encounter frequency, ν, and a factor (much smaller than 1) that expresses the fact that only a small proportion of encounters lead to the formation or

dissolution of a link. From now on, we shall denote that product, which might be called the *effective encounter frequency*, by ν, and as it is proportional to the number of elements in the system, write it as $\nu = \nu_0 \cdot n$.

A consequence of this more detailed view of the formation and dissolution processes is that the parameter φ is no longer an input parameter, but becomes a function of the new parameter, the system temperature. With the net rate of link formation again denoted by μ, then, with reference to Sec. 1.2,

$$\mu = \Phi(\theta;\varepsilon_1)(m-\chi)^2 - \Phi(\theta;\varepsilon_2)2\chi \ , \tag{4.9}$$

resulting in

$$\chi = m + \varphi - (\varphi^2 + 2m\varphi + \frac{\mu}{\Phi(\theta;\varepsilon_1)})^{\frac{1}{2}} \ . \tag{4.10}$$

For the particular picture of the internal activities of the system in terms of a system temperature, Eq. (4.10) is now the explicit form of the coupling between the internal and external processes.

There remains to define the dependence of λ on θ, and in the absence of any contrary information, it is natural to assume that the Arrhenius relation holds,

$$\lambda = \lambda_0 e^{\frac{\kappa\theta}{\varepsilon_3}} \ , \tag{4.11}$$

and ε_3 is the *failure activation energy*.

4.3 Two Simple Cases

To see where this leads, consider first the case where $f(\alpha;\theta) = \delta(\alpha - \kappa\theta)$. Then

$$\Phi(\theta;\varepsilon) = \begin{cases} 0; \ \kappa\theta < \dfrac{\varepsilon}{2} \\ \nu(2\kappa\theta - \varepsilon); \ \kappa\theta > \dfrac{\varepsilon}{2} \end{cases} \tag{4.12}$$

Next, let $f(\alpha;\theta)$ be a simple, square-shaped function,

$$f(\alpha;\theta) = \begin{cases} \dfrac{1}{2\beta}; \ \kappa\theta - \beta < \alpha < \kappa\theta + \beta \\ 0; \ \textit{otherwise} \end{cases} \tag{4.13}$$

In this case it is convenient to introduce the new dimensionless variable, ϑ,

$$\vartheta = \frac{\kappa\theta - \dfrac{\varepsilon}{2}}{\beta} \ ; \tag{4.14}$$

with which Φ(ϑ;ε) is given by

$$\Phi(\vartheta;\beta)=\begin{cases}0;\ \vartheta<-1\\ \dfrac{\nu\beta}{3}(\vartheta+1)^3\ ;-1<\vartheta<0\\ \nu\beta(\vartheta-\dfrac{2}{3}(1-\vartheta)^3+1)\ ;0<\vartheta<1\\ 2\nu\beta\vartheta\ ;\vartheta>1\end{cases} \tag{4.15}$$

For this particular choice of $\Phi(\theta;\varepsilon)$, Fig. 4.1 shows $\Phi(\theta;\varepsilon_1)$ and $\Phi(\theta;\varepsilon_2)$, as well as their ratio, φ, which was defined in Eq. (1.4), and this second case will now be used to carry out a set of numerical simulations.

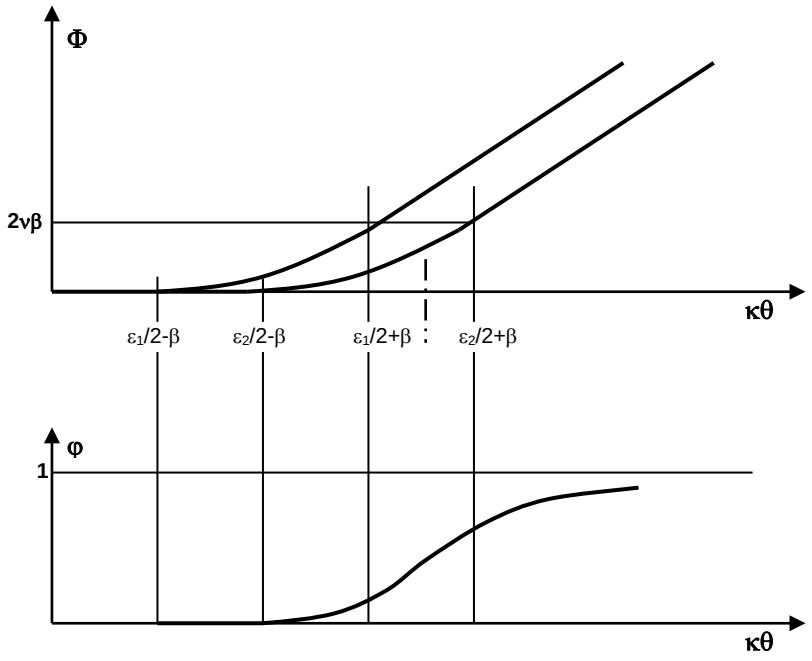


Figure 5.1 The two proportionality factors, $\Phi(\theta;\varepsilon_1)$ and $\Phi(\theta;\varepsilon_2)$, and their ratio, φ, as defined by Eq. (2.4).

4.4 Numerical Simulations

The same simulation program as used in Section 3, modified to introduce the system temperature as a parameter in accordance with Eq. (4.14), was used to simulate the behaviour of the system and to determine the average life time as a function of temperature. The base case is now given by the following parameter values:

$m_0 = 4$, $g_0 = 0.5$, $\alpha = 0.6$, $\lambda_0 = 0.005$, $\varepsilon_1 = 1.8$, $\varepsilon_2 = 2.2$, $\varepsilon_3 = 2.0$, $\beta = 0.1$, and $\nu_0 = 1$.

Figure 4.2 shows the life time, and Fig. 4.3 shows the system size (i.e. the value of *n* during the survival phase), both as functions of system temperature (actually, as a function of κT). As the temperature increases above 1.24, the lifetime decreases rapidly, and as the temperature decreases below this value, the system size decreases rapidly. Therefore, there is only a narrow range of system temperature around 1.24 where the system is viable in the given environment.

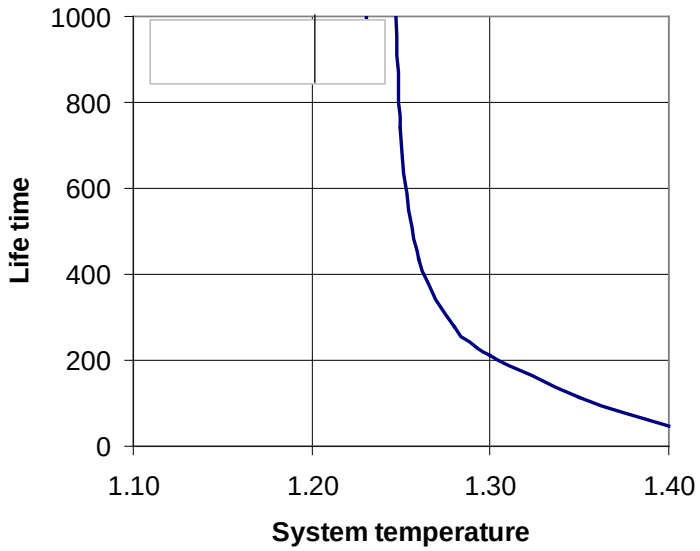


Figure 4.2 The life time as a function of the system temperature.

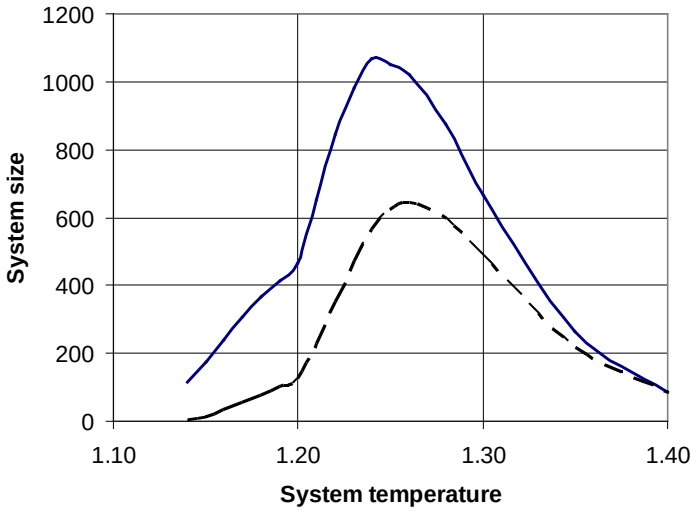


Figure 4.3 The system size as a function of system temperature. The full line is the size of the system during the survival phase; the dashed line is the size the system would reach (and remain in) were there no fluctuations.

Of course, the curves shown in these two Figures result from the particular choice we made for the relationship between system temperature and the probabilities for formation and dissolution of links as well as the port failure rate. However, in general we would expect a limited range of existence for the system, bounded by the increase in failure rate with increasing temperature and the decrease in the ordering activity (through the encounters) with decreasing temperature. This behaviour was foreshadowed in (Aslaksen, 2004), where an engineering organisation was used as an example to demonstrate the importance of a balance between freedom (fluctuations leading to innovation, but also to organisational breakdown) and conformance (leading to efficiency).

5 CONCLUSION

We have investigated the dynamic behaviour of a relatively simple system that has many of the basic characteristics of "living" systems, i.e. systems that are able to maintain themselves by interaction with their environment. Foremost among these is the characteristic that the success of the system in doing so depends on the adaptation of the system to the environment, in the sense that the complexity of the system in relation to the complexity of the environment. In our system, complexity was represented by the structure, and only the very simplest measure of this complexity was introduced.

To examine the dynamic behaviour, a numerical model of the system was developed in the form of a Visual Basic program. The most striking outcome of running that program is the influence of fluctuations; without fluctuations the system grows to a certain size and then remains in that state indefinitely, but fluctuations can lead to instability of that state and thereby a finite life time, as well as influence the growth phase in both prolonging it and increasing the system size. In other words, this is an example of where the fluctuations drive the average, as discussed e.g. in (Nicolis, 1977).

In analogy with statistical mechanics, we proposed a possible relationship between the stochastic processes of decay and creation (or maintenance) and a quantity that can be interpreted as a system temperature; a measure of the activity level of the system, both within and in its interactions with its environment. The particular relationship proposed demonstrates that such a system is only viable within a very narrow, also has interesting implications for engineered systems, as was discussed previously (Aslaksen, 2004).

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